Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-25 (2015) 162–165

One estimation of the stability defect of sets in an approach game problem ⋆

V.N. Ushakov * **A.G. Malev** **

* *Institute of Mathematics and Mechanics named after N.N. Krasovskii, Ural Branch of the Russian Academy of Sciences, S. Kovalevskaya str. 16, Yekaterinburg, 620990 Russia, and Ural Federal University named after B.N. Yeltsin, Mira str. 19 , Yekaterinburg, 620002 Russia (e-mail: ushak@imm.uran.ru)*
** *Institute of Mathematics and Mechanics named after N.N. Krasovskii, Ural Branch of the Russian Academy of Sciences, S. Kovalevskaya str. 16, Yekaterinburg, 620990 Russia (e-mail: malev.alexey@gmail.com)*

Abstract: A game problem of the approach to a compact target set at a fixed termination time is studied. We investigate the question of estimating the stability defect of a set in the space of game positions, which is weakly invariant with respect to a finite set of unification differential inclusions.

Keywords: Differential games, control theory, stable bridge, stability defect.

1. INTRODUCTION

We consider a conflict-control nonlinear system on a finite time interval and study a game problem of approach of the system to a target compact set at a fixed time. The main subject of the research is the notion of the stability defect of sets in the space of game positions of the system, earlier introduced and investigated by Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011). This notion was introduced in order to extend the stability property and due to the fact that quite often in the process of constructing stable bridges one can get sets not possessing the stability property. The stability property is a property of weak invariance of a set in the space of positions with respect to some set of differential inclusions related to the dynamics of the system. These sets of inclusions can be different, but they identify the same sets in the space of game positions, which are stable bridges. The formulation of the stability property based on unification turned out to be convenient for extending the stability property. It was proposed in researches of Krasovskii (1976), and Krasovskii (1977). In particular, unification definitions of the stability property presented in the infinitesimal form appeared to be effective, see Guseinov, Subbotin, and Ushakov (1985)

We note that unification sets of differential inclusions, expressed in unification definitions of the stability property and used in this paper, are infinite. For any non-trivial system it is impossible to realistically check, whether a set in the space of positions is stable bridge. This check can be carried out for some relatively simple conflict-control systems, e.g. for systems having a simple Hamiltonian, by the fact that the unification set can be replaced with a finite subset being equivalent in terms of the stability property.

For arbitrary conflict-control systems with complicated dynamics following problem becomes relevant. Suppose that one has selected a finite subset of unification set of differential inclusions. Furthermore, there is a set constructed in the space of positions, that is weakly invariant with respect to the subset. It is required to assess to what extent does the set possess the stability property, i.e. in what extent is it weakly invariant with respect to the whole unification set. In other words, it is required to estimate an upper bound of value of the stability defect of the set. Current paper is dedicated to derivation of one of these assessments.

2. SETTING OF THE APPROACH GAME PROBLEM

Let us consider a conflict-control system whose behavior on the time interval $[t_0, \vartheta]$ $(t_0 < \vartheta < \infty)$ is described by the differential equation

$$\frac{dx}{dt} = f(t, x, u, v), \quad x(t_0) = x^0, \quad u \in P, \quad v \in Q. \quad (1)$$

Here, x is the m-dimensional state vector of the system; u is the control of the first player; v is the control of the second player; and P and Q are compact sets in the Euclidean spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

We assume that the right-hand side of equation (1) satisfies the following conditions.

Condition A. The function $f(t, x, u, v)$ is given and continuous on $[t_0, \vartheta] \times \mathbb{R}^m \times P \times Q$ and, for any bounded closed domain $D \subset [t_0, \vartheta] \times \mathbb{R}^m$, there exists a constant $L = L(D) \in (0, \infty)$ such that

⋆ This work was supported by the Russian Science Foundation (project no. 15-11-10018).

2405-8963 © 2015, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.11.077

$$\|f(t,x^{(1)},u,v) - f(t,x^{(2)},u,v)\| \leqslant L\|x^{(1)} - x^{(2)}\|,$$
$$(t,x^{(i)},u,v) \in D \times P \times Q, \quad i = 1,2.$$

Here, $\|f\|$ stands for the norm of the vector f in the Euclidean space.

Condition B. There exists a constant $\gamma \in (0,\infty)$ such that

$$\|f(t,x,u,v)\| \leqslant \gamma(1 + \|x\|),$$
$$(t,x,u,v) \in [t_0,\vartheta] \times \mathbb{R}^m \times P \times Q.$$

In the approach game problem facing the first player, it is required to ensure that the phase vector $x(\vartheta)$ of equation (1) hits a given compact set $M \subset \mathbb{R}^m$. The solution of the approach problem is required to be in the class of positional control procedures of the first player, see Krasovskii, and Subbotin (1974).

The dual problem to the problem formulated above, is an evasion problem facing the second player, in which it is required to ensure the evasion of the phase vector $x(\vartheta)$ of equation (1) from some ε-neighborhood of the compact set M. The solution of the evasion problem is required to be in the class of counter-positional control procedures with a guide of the second player, see Krasovskii, and Subbotin (1974).

A differential approach-evasion game is composed of the approach problem and the evasion problem. According to Guseinov, Subbotin, and Ushakov (1985), the game has the following alternative: there exists a closed set $W^0 \subset [t_0,\vartheta] \times \mathbb{R}^m$ such that the approach problem is solvable for all initial positions $(t_*,x_*) \in W^0$ and the evasion problem is solvable for all initial positions $(t_*,x_*) \in \big([t_0,\vartheta] \times \mathbb{R}^m\big) \setminus W^0$. The set W^0 plays a decisive role in solving the differential approach-evasion game. For initial positions $(t_*,x_*) \in W^0$ a resolving positional control procedure of the first player can be implemented as a positional control procedure with a guide, that directs the phase vector $x(t)$ of equation (1) towards the guide moving in the set W^0. It is known that the set W^0 has following very important property: W^0 is the maximal u-stable bridge, see Krasovskii, and Subbotin (1974). This property forms the basis of algorithms for the approximate calculation of W^0. Some studies use algorithms for the approximate calculation of W^0 based on unification constructions or their modifications, see Tarasev, and Ushakov (1987) and Tarasev, Ushakov, and Khripunov (1987).

Let us formulate a definition of the stability property of sets contained in $[t_0,\vartheta] \times \mathbb{R}^m$ based on unification set of differential inclusions in the infinitesimal form. To describe the unification set we introduce Hamiltonian function H of equation (1) and a set of multivalued mappings $\mathcal{L}$.

Assume that

$$H(t,x,l) = \max_{u \in P} \min_{v \in Q} \langle l, f(t,x,u,v)\rangle,$$
$$(t,x,l) \in [t_0,\vartheta] \times \mathbb{R}^m \times \mathbb{R}^m,$$

is Hamiltonian function of equation (1), where $\langle l, f\rangle$ is the inner product of the vectors l and f from $\mathbb{R}^m$.

Taking into account Condition B and the definition of the set W^0, we conclude that one can find a sufficiently large bounded closed domain D in $[t_0,\vartheta] \times \mathbb{R}^m$ that contains the set W^0 and all motions $x(t)$ coming to some ε-neighborhood of the set M, i.e. $(t,x(t)) \in D$ when $t \in [t_0,\vartheta]$. Hereinafter D is fixed.

We choose $R \in (0,\infty)$ to be so large that

$$r = \max_{(t,x,l) \in D \times S} |H(t,x,l)| < R,$$

where $S = \big\{l \in \mathbb{R}^m : \|l\| = 1\big\}$.

We follow Krasovskii (1976) and Krasovskii (1977), and introduce following sets:

$$G = B(\mathbf{0};R), \quad \Pi_l(t,x) = \big\{f \in \mathbb{R}^m\colon \langle l,f\rangle \leqslant H(t,x,l)\big\},$$
$$F_l(t,x) = \Pi_l(t,x) \cap G, (t,x,l) \in D \times S.$$

Here $B(\mathbf{0};R)$ is the closed ball in $\mathbb{R}^m$ with center at $\mathbf{0}$ and radius R.

Thus, the sets $F_l(t,x)$ are the spherical segments in the space $\mathbb{R}^m$, that don't degenerate for any $(t,x,l) \in D \times S$, i.e. sets have a non-empty interior.

Let us define the set $\mathcal{L}$ as a family of multivalued mappings $(t,x) \mapsto F_l(t,x)$ defined on D and corresponding to vectors $l \in S$. Obviously, the set $\mathcal{L}$ is uncountable.

We consider

$$\overrightarrow{D}W(t_*,x_*) = \Big\{d \in \mathbb{R}^m\colon d = \lim_{k\to\infty}(t_k - t_*)^{-1}(w_k - x_*),$$
$$\{(t_k,w_k)\} \text{ is a sequence in } W,$$
$$t_k \downarrow t_* \text{ when } k \to \infty,\ \lim_{k\to\infty} w_k = x_*\Big\}.$$

$\overrightarrow{D}W(t_*,x_*)$ is the contingent derivative of the multivalued mapping $t \mapsto W(t) = \big\{x \in \mathbb{R}^m : (t,x) \in W\big\}$ at the point $(t_*,x_*) \in W$, $t_* \in [t_0,\vartheta)$, see Guseinov, Subbotin, and Ushakov (1985).

Definition 1. A non-empty closed set $W \subset D$ is a u-stable bridge in the approach game problem at time ϑ if and only if:

(1) $W(\vartheta) \subseteq M$;
(2) $\overrightarrow{D}W(t_*,x_*) \cap F_l(t_*,x_*) \neq \varnothing, \quad t_* \in [t_0,\vartheta),$
$(t_*,x_*,l) \in \partial W \times S.$

The definition 1 is the definition of the stability property in the infinitesimal form. It embeds the notion of derivatives into theory of differential games. The definition turned out to be useful in identifying the various properties of stable bridges, see Ushakov et al. (2010). Additionally, it is helpful in the formation of new concepts and constructions in the theory of differential games, see Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011).

3. STABILITY DEFECT OF SETS IN THE SPACE OF GAME POSITIONS

In this section, we give a definition of the stability defect of a set $W^* \subset D$. We assume that $W^*(\vartheta) = M$ and W^* has the continuity property: if $t_0 \leqslant t_* < t^* \leqslant \vartheta$ and $W^*(t_*) \neq \varnothing$ then $W^*(t^*) \neq \varnothing$.

Moreover, strengthening the continuity property of the set W^*, we assume that the following condition is satisfied.

Condition C.

$$d\big(W^*(t_*), W^*(t^*)\big) \leqslant R(t^* - t_*), \quad t_0 \leqslant t_* < t^* \leqslant \vartheta.$$

Here, $d(W_1, W_2)$ is the Hausdorff distance between the compact sets W_1 and W_2 in the space $\mathbb{R}^m$.

Condition C means that the multivalued mapping $t \mapsto W^*(t), t \in [t_0, \vartheta]$, is Lipschitz with Lipschitz constant $R \in (0, \infty)$. Condition C is not too restrictive for W^*. It implies that the set W^* satisfies the relation

$$\overrightarrow{D}W^*(t_*, x_*) \cap G \neq \varnothing, \quad (t_*, x_*) \in \partial W^*, \quad t_* \in [t_0, \vartheta).$$

This condition is similar to the condition

$$\overrightarrow{D}W^0(t_*, x_*) \cap G \neq \varnothing, \quad (t_*, x_*) \in \partial W^0, \quad t_* \in [t_0, \vartheta),$$

which is valid for the maximal u-stable bridge W^0 in the approach problem.

With every point $(t_*, x_*) \in \partial W^*$, $t_* \in [t_0, \vartheta)$, we associate the number

$$\varepsilon(t_*, x_*) = \sup_{l \in S} \rho\big(\overrightarrow{D}W^*(t_*, x_*), F_l(t_*, x_*)\big).$$

Here, $\rho(W_1, W_2) = \inf\big\{\|w_1 - w_2\| : (w_1, w_2) \in W_1 \times W_2\big\}$.

The value $\varepsilon(t_*, x_*)$ is called the *stability defect* of the set W^* at the point $(t_*, x_*) \in \partial W^*$, $t_* \in [t_0, \vartheta)$, see Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011).

The value $\varepsilon(t_*, x_*) \geqslant 0$ is a local characteristic of the degree of instability of the set W^* at the point $(t_*, x_*) \in \partial W^*$. It shows, how much the dynamics of a conflict-control system isn't aligned with the evolution of the set W^* in time, from the stability property point of view. A great value $\varepsilon(t_*, x_*)$ means strong misalignment between the dynamics of the system and the set W^*. Equality $\varepsilon(t_*, x_*) = 0$ means that W^* possesses the stability property at point (t_*, x_*).

For $t_* \in [t_0, \vartheta)$, we set

$$\varepsilon(t_*) = \sup_{(t_*, x_*) \in \Lambda(t_*)} \varepsilon(t_*, x_*),$$

where $\Lambda(t_*) = \partial W^* \cap \Gamma_{t_*}$ and $\Gamma_{t_*} = \big\{(t, x) : t = t_*\big\}$. Additionally, we extend this definition of $\varepsilon(t_*)$ by setting $\varepsilon(\vartheta) = 0$. Thus, we get a non-negative function $\varepsilon(t)$ defined on segment $[t_0, \vartheta]$.

The value $\varepsilon(t_*)$ is called the *stability defect* of the set W^* at time $t_* \in [t_0, \vartheta]$. It's noted by Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011) that $\varepsilon(t) \equiv 0$ if and only if W^* is a u-stable bridge. Therefore, if $\varepsilon(t) \equiv 0$ on $[t_0, \vartheta]$, then the rule of extremal aiming at a guide moving along W^* guarantees that the phase vector $x(\vartheta)$ of the equation (1) hits M in the case $(t_*, x(t_*)) \in W^*$.

We note that under Condition A the set-valued mapping $(t, x, l) \mapsto F_l(t, x)$ is continuous (in the Hausdorff metric) on the compact set $D \times S$. Hence, $(t, x, l) \mapsto F_l(t, x)$ is uniformly continuous on $D \times S$. Also it's not hard to show that there exists a constant $\lambda = \lambda(L) \in (0, \infty)$ such that

$$d\big(F_l(t, x^{(1)}), F_l(t, x^{(2)})\big) \leqslant \lambda \left\|x^{(1)} - x^{(2)}\right\|,$$
$$l \in S, \quad (t, x^{(i)}) \in D, \quad i = 1, 2.$$

A formula for calculating λ was mentioned in the research of Ushakov (1980) and in terms of the current paper it is $\lambda = \lambda(L) = \dfrac{RL}{\sqrt{R^2 - r^2}}$.

Under certain additional conditions on W^* and when the function $\varepsilon(t)$ on $[t_0, \vartheta]$ is small then the rule of extremal aiming at a guide moving along W^* guarantees that the phase vector $x(\vartheta)$ of the equation (1) hits a small ε-neighborhood of the target set M, see Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011). It is also shown that the value ε can be expressed via $\varepsilon(t)$, $t \in [t_0, \vartheta]$. Below we explain in more detail what we just mentioned.

In addition to condition C we impose following conditions on the set W^* and the function $\varepsilon(t)$.

Condition D. There exists a function $\varphi^*(\delta) \geqslant 0$ on $(0, \vartheta - t_0)$ ($\varphi^*(\delta) \downarrow 0$ as $\delta \downarrow 0$) such that

$$h\left(\overrightarrow{D}^{\nabla} W^*(t_*, x_*), \delta^{-1}\big(W^*(t_* + \delta) - x_*\big)\right) \leqslant \varphi^*(\delta),$$
$$(t_*, x_*) \in \partial W^*, \quad t_* \in [t_0, \vartheta), \quad \delta \in (0, \vartheta - t_*).$$

Here $\delta^{-1}\big(W^*(t_* + \delta) - x_*\big) = \big\{f \in \mathbb{R}^m : f = \delta^{-1}(w^{(\delta)} - x_*),\ w^{(\delta)} \in W^*(t_* + \delta)\big\}$, $h(W_1, W_2)$ is the Hausdorff deviation of the compact set W_1 from the compact set W_2 in $\mathbb{R}^m$, $\overrightarrow{D}^{\nabla} W^*(t_*, x_*) = \overrightarrow{D}W^*(t_*, x_*) \cap B(\mathbf{0}; 3R)$.

Condition E. The function $\varepsilon(t)$ is Lebesgue measurable on $[t_0, \vartheta]$.

Let us introduce a set $\mathcal{W}^* \subset [t_0, \vartheta] \times \mathbb{R}^m$:

$$\mathcal{W}^*(t) = W^*(t) + B(\mathbf{0}; \varkappa(t)),\ t \in [t_0, \vartheta],$$

where $\varkappa(t) = \displaystyle\int_{t_0}^{t} \mathrm{e}^{\lambda(t-\tau)} \varepsilon(\tau) \mathrm{d}\tau$ is the Lebesgue integral.

The set $\mathcal{W}^*$ satisfies the initial condition $\mathcal{W}^*(t_0) = W^*(t_0)$, and its cross sections $\mathcal{W}^*(t)$ are $\varkappa(t)$-neighborhoods of cross sections $W^*(t)$ in $\mathbb{R}^m$. $\varkappa(t)$ increases monotonically with increasing t.

Theorem 2. $\mathcal{W}^*$ is a u-stable bridge in the approach game problem of the equation (1) at time ϑ with a set $M_{\varkappa(\vartheta)}$.

The proof of the theorem one can find in the paper of Ushakov, and Malev (2011). The value $\varepsilon_{W^*} = \varkappa(\vartheta)$ is called the *stability defect* of the set W^*.

4. ESTIMATION OF THE STABILITY DEFECT OF WEAKLY INVARIANT SET

After we carefully defined the concept of the stability defect ε_{W^*}, we can consider various sets W^* for the calculation or estimation of the stability defect ε_{W^*}. If the value of stability defect ε_{W^*} becomes small for some specific approach game problem and a set W^*, then it makes sense to consider the weak approach game problem with a target set $M_{\varepsilon_{W^*}}$.

It makes sense to consider a variety of notable sets $W^* \subset [t_0, \vartheta] \times \mathbb{R}^m$ satisfying the equation $W^*(\vartheta) = M$. A great interest is to assess to what extent these sets are stable bridges in the approach game problem with M at time ϑ. Program absorption set and positional absorption set in the approach game problem are good examples, see Krasovskii, and Subbotin (1974). The assessment of the question is reduced to calculating the values or finding the upper bounds for the values of the stability defect for such sets. In this paper, we examine one class of sets $W^* \subset D$

and obtain an upper bound for the value of the stability defect ε_{W^*}.

We consider a finite subset $\mathcal{L}^* = \{(t,x) \mapsto F_{l_\rho}(t,x), \rho \in \overline{1,N}\}$ of set of multivalued mappings $\mathcal{L} : (t,x) \mapsto F_l(t,x)$. We allocate the maximal (by inclusion) set W^* ($W^*(\vartheta) = M$) in the domain D, weakly invariant with respect to the finite set of differential inclusions $\mathcal{N}^*$ corresponding to mappings $\mathcal{L}^*$:

$$\frac{dx}{dt} \in F_{l_\rho}(t,x), \quad \rho \in \overline{1,N}, \quad t \in [t_0, \vartheta]. \tag{2}$$

It is clear that W^* is a non-empty closed set in D satisfying the inclusion $W^0 \subseteq W^*$. In some rare cases it may happen that $W^0 = W^*$. But in the vast majority of cases there is a strict inclusion $W^0 \subset W^*$ and, therefore, in these cases the set W^* does not possess the stability property. Naturally, a question arises to what extent the W^* is unstable. So, our task is to obtain an upper bound for the value of the stability defect of the set W^*. For this assessment we present some auxiliary assertions studied and proven in detail in the research of Malev (2013).

Lemma 3. There exists a constant $L_H \in (0, \infty)$ that Hamiltonian function of equation (1) satisfies the inequality

$$|H(t,x,l^*) - H(t,x,l_*)| \leqslant L_H \|l^* - l_*\|, \\ (t,x) \in D, \quad l_*, l^* \in S.$$

One can assume $L_H = \max\limits_{(t,x,u,v) \in D \times P \times Q} \big\| f(t,x,u,v) \big\|$, see Malev (2013).

Lemma 4. There exists a constant $L_F \in (0, \infty)$ that

$$d\big(F_{l_*}(t,x), F_{l^*}(t,x)\big) \leqslant L_F \|l_* - l^*\|, \\ (t,x) \in D, \quad l_*, l^* \in S, \quad |l_* - l^*| \leqslant \xi.$$

Here $\xi \in (0, 1 - r/R)$ is a preselected, typically small value.

The constant L_F can be defined by means of the formula $L_F = \dfrac{R(L_H + R)}{\sqrt{R^2 - r^2}}$, see Malev (2013).

Theorem 5. There is a finite set $S^{(\delta)} = \{l_\rho : \rho \in \overline{1,N}\} \subset S$ which is δ-net of the sphere $S \subset \mathbb{R}^m$, where $\delta \in (0, \xi]$. Then the stability defect ε_{W^*} of the maximal weakly invariant set W^* *($W^*(\vartheta) = M$)*, with respect to the finite set of differential inclusions $\mathcal{N}^*$, satisfies inequality

$$\varepsilon_{W^*} \leqslant L_F \frac{e^{\lambda(\vartheta - t_0)} - 1}{\lambda} \delta.$$

The theorem 5 is the central statement of the paper. One can find a proof in the research of Malev (2013). The theorem presents the estimation of the stability defect of the set weakly invariant with respect to the finite set of differential inclusions $\mathcal{N}^*$ based on the finite δ-net of the unit sphere in $\mathbb{R}^m$.

REFERENCES

H.G. Guseinov, A.I. Subbotin, and V.N. Ushakov. Derivatives for multivalued mappings with applications to game-theoretical problems of control. *Problems Control Inform. Theory*, volume 14, pages 405–419, 1985.

N.N. Krasovskii. *Dokl. Akad. Nauk SSSR*, volume 226, suppl. 6, pages 1260–1263, 1976.

N.N. Krasovskii. *Inst. Mat. Mekh. Uralsk. Nauchn. Tsentra Akad. Nauk SSSR*, volume 24: Game Control Problems, pages 32–45, Sverdlovsk, 1977.

N.N. Krasovskii, and A.I. Subbotin. Positional Differential Games. *Nauka*, Moscow, 1974.

A.G. Malev. *Trudy Inst. Mat. Mekh. UrO RAN*, volume 19, suppl. 1, pages 205–216, 2013.

A.M. Tarasev, and V.N. Ushakov. On Constructing Stable Bridges in a Minimax Game of Approach-Evasion. *Available from VINITI*, No. 454-83, Sverdlovsk, 1983.

A.M. Tarasev, V.N. Ushakov, and A.P. Khripunov. *J. Appl. Math. Mech.*, volume 51, suppl. 2, pages 216–222, 1987.

V.N. Ushakov. *Inst. Mat. Mekh. Uralsk. Nauchn. Tsentra Akad. Nauk SSSR*, No. 4425-80, Sverdlovsk, 1980.

V.N. Ushakov, H.G. Guseinov, Ya.A. Latushkin, and P.D. Lebedev. On the coincidence of maximal stable bridges in two approach game problems for stationary control systems *Proc. Steklov Inst. Math.*, volume 268, suppl. 1, pages 240–263, 2010.

V.N. Ushakov, and Ya.A. Latushkin. The stability defect of sets in game control problems. *Proc. Steklov Inst. Math.*, volume 255, suppl. 2, pages 198–215, 2006.

V.N. Ushakov and A.G. Malev. On the question of the stability defect of sets in an approach game problem. *Proc. Steklov Inst. Math.*, volume 272, suppl. 1, pages 229–254, 2011.

V.N. Ushakov, and A.A. Uspenskii. On a supplement to the stability property in differential games. *Proc. Steklov Inst. Math.*, volume 271, suppl. 1, pages 286–305, 2010.